— * Vector Integration * —

Introduction :- Integration is an inverse operation of differentiation.

Let '$\bar{F}(t)$' be a differentiable vector f^n. with scalar variable 't' then

$$\frac{d}{dt}[\bar{F}(t)] = \bar{f}(t)$$

$$\rightarrow \boxed{\bar{F}(t) = \int \bar{f}(t)\, dt}$$

Here '$\bar{F}(t)$' is called "Premitive of $\bar{f}(t)$" (also integration)

In general $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ is any vector then

$$\bar{F}(t) = \int\{f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}\}\, dt$$

$$\therefore \boxed{\bar{F}(t) = \bar{i}\int f_1(t)\, dt + \bar{j}\int f_2(t)\, dt + \bar{k}\int f_3(t)\, dt}$$

Definate integral :-

$$\bar{F}(t) = \int_a^b [f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}]\, dt$$

$$= \bar{i}\int_a^b f_1(t)\, dt + \bar{j}\int_a^b f_2(t)\, dt + \bar{k}\int_a^b f_3(t)\, dt$$

Workdone by a force :- If '$\bar{F}$' represents the force vector along the particle AB, then the workdone during the small displacement $\delta\bar{r}$ is given by $\int_A^B \bar{F}\cdot d\bar{r}$

Defination of line Integral :- $\bar{F}(r)$ is a continuous vector f^n defined on the curve 'c' with the tangent component is $\int_c \bar{F}(r)\cdot d\bar{r}$ is called "line integral."

eg :- Find $\int_c \bar{F}\cdot d\bar{r}$ where $\bar{F} = xy\bar{i} + yz\bar{j} + zx\bar{k}$ and the curve 'c' is $\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$ where t is from '-1' to '$+1$'.

Sol^n: Given $\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$ & $\bar{F} = xy\bar{i} + yz\bar{j} + zx\bar{k}$

then $x = t,\ y = t^2,\ z = t^3$.

$d\bar{r} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}$ $\quad$ $\bar{F} = z\bar{i} + yz\bar{j} + zx\bar{k}$

$\bar{F} = t^3\bar{i} + t^5\bar{j} + t^4\bar{k}$

Now $\bar{F}\cdot d\bar{r} = (t^3\bar{i} + t^5\bar{j} + t^4\bar{k})\cdot(\bar{i} + 2t\bar{j} + 3t^2\bar{k})$

$= t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$

$\therefore \int_{-1}^{1} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (t^3 + 5t^6)\,dt = \left(\frac{t^4}{4} + \frac{5t^7}{7}\right)_{-1}^{1} = \left(\frac{1}{4} + \frac{5}{7}\right) - \left(\frac{1}{4} - \frac{5}{7}\right)$

$= \frac{10}{7}$ //

Problem :-

(1). Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the st. line from $(0,0,0)$ to $(2,1,3)$.

Sol:- the eqⁿ of st. line joining points $(0,0,0)$ to $(2,1,3)$ is
(x_1, y_1, z_1) $\quad$ (x_2, y_2, z_2)

$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \Rightarrow \frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0}$

$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say)

$\Rightarrow x = 2t \;;\; y = t \;;\; z = 3t$

$\therefore (x, y, z) = (2t, t, 3t)$

& $t = 0 \rightarrow t = 1$

$\therefore$ workdone in moving along the line from $O(0,0,0)$, $A(2,1,3)$

is $\int_C \bar{F}\cdot d\bar{r}$ ——— (1)

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{r} = 2t\bar{i} + t\bar{j} + 3t\bar{k}$

$d\bar{r} = 2\bar{i} + \bar{j} + 3\bar{k}$

& $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

$= 12t^2\bar{i} + (12t^2 - t)\bar{j} + (3t)\bar{k}$

$\therefore \bar{F}\cdot d\bar{r} = 24t^2 + 12t^2 - t + 9t = 36t^2 + 8t$

$(1) \Rightarrow \int_C \bar{F}\cdot d\bar{r} = \int_0^1 (36t^2+8t)\,dt = \left(36\frac{t^3}{3}+8\frac{t^2}{2}\right)_0^1 = 12+4 = 16$ //

② Find the workdone by the force $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$ when it moves a particle from the point $(0,0,0)$ to $(2,1,1)$ along the curve $x=2t^2$, $y=t$ and $z=t^3$.

Sol:- Given $(x,y,z) = (2t^2, t, t^3)$

$t=0 \rightarrow t=1$

Workdone $= \int_C \bar{F}\cdot d\bar{r}$ —— (1)

Here $\bar{F} = (2t+3)\bar{i} + (2t^5)\bar{j} + (t^4-2t^2)\bar{k}$

& $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$\bar{r} = 2t^2\bar{i}+t\bar{j}+t^3\bar{k}$

$d\bar{r} = 4t\bar{i}+\bar{j}+3t^2\bar{k}$

$(1) \rightarrow = \int_0^1 \left((8t^2+12t)+2t^5+3t^6-6t^4\right)dt$

$= \left(8\frac{t^3}{3}+\frac{12t^2}{2}+\frac{2t^6}{6}+\frac{3t^7}{7}-\frac{6t^5}{5}\right)_0^1$

$= \frac{8}{3}+6+\frac{1}{3}+\frac{3}{7}-\frac{6}{5} = \frac{\cancel{8}35+105+35+45-126}{105} = \frac{288}{35}$ //

③ If $\bar{F} = (5xy-6x^2)\bar{i} + (2y-4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve 'C' in xy-plane $y=x^3$ from $(1,1)$ to $(2,8)$.

Sol: $\bar{F} = (5x^4-6x^2)\bar{i} + (2x^3-4x)\bar{j}$.

Along the curve, $y=x^3 \Rightarrow dy = 3x^2dx$

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$d\bar{r} = dx\bar{i}+dy\bar{j}+dz\bar{k}$

$= dx\bar{i}+3x^2dx\bar{j}+0\bar{k}$

$$\therefore \bar{F}\cdot d\bar{r} = \left[(5x^4-6x^2)\bar{i} + (2x^3-4x)\bar{j}\right]\cdot\left(dx\,\bar{i} + 3x^2dx\,\bar{j}\right)$$

$$= (5x^4-6x^2)\,dx + (2x^3-4x)\,3x^2dx$$

$$= \left(5x^4-6x^2+6x^5-12x^3\right)dx$$

Hence

$$\oint_C \bar{F}\cdot d\bar{r} = \int_1^2 \left(5x^4-6x^2+6x^5-12x^3\right)dx$$

$$= \left(\frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4}\right)_1^2 = 35 \; //$$

(4). Evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ over arc of a helix $x = a\cos t$, $y = a\sin t$, $z = kt$ as 't' varies from $0 \to 2\pi$

Sol^n:- Given

$x = a\cos t$	$y = a\sin t$	$z = kt$
$dx = -a\sin t\,dt$	$dy = a\cos t\,dt$	$dz = k\,dt$

$$\therefore \oint_C (yz\,dx + xz\,dy + xy\,dz) = \oint_0^{2\pi} a\sin t\,(kt)\left(-a\sin t\,dt\right) + a\cos t\,(kt)\,a\cos t\,dt + (a^2\cos t)\sin t\;k\,dt$$

$$= \int_0^{2\pi} -a^2kt\sin^2 t\,dt + a^2\cos^2 t\,kt\,dt + a^2\cos t\sin t\,k\,dt$$

$$= a^2k\int_0^{2\pi}\left(-t\sin^2 t + t\cos^2 t\,dt\right) + \frac{a^2k}{2}\left(2\sin t\cos t\right)dt$$

$$= a^2k\int_0^{2\pi} t\left(\cos^2 t - \sin^2 t\right)dt + \frac{a^2k}{2}\int_0^{2\pi}\left(\sin 2t\right)dt$$

$$= a^2k\int_0^{2\pi} t\cos 2t\,dt + \frac{a^2k}{2}\int_0^{2\pi}\sin 2t\,dt$$

$$= a^2k\left[\left(t\,\frac{\sin 2t}{2}\right)_0^{2\pi} - \int_0^{2\pi}\frac{\sin 2t}{2}\,dt\right] + \frac{a^2k}{2}\left(-\frac{\cos 2t}{2}\right)_0^{2\pi}$$

$$= a^2k\left[0 + \frac{1}{4}\left(\cos 2t\right)_0^{2\pi}\right] + \frac{a^2k}{4}\left[-\cos 4\pi + 1\right]$$

$$= a^2k\left[\frac{1}{4}\left(\cancel{\cos 4\pi - 1}\right)\right] + \frac{a^2k}{4}\left[\cancel{1-\cos 4\pi}\right]$$

$$= 0 \; //$$

3). Find the workdone (or) Circulation by $\bar{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$ along a curve C in the xy-plane given by

(i). $x^2+y^2=9$, $z=0$

$\int_C \bar{F}\cdot d\bar{r}$ = workdone (Circulation)

ii). $x^2+y^2=4$, $z=0$.

Sol^n:- Given $\bar{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$

In the xy-plane $z=0$ $\therefore dz=0$

$\rightarrow$ $\bar{F} = (2x-y)\bar{i} + (x+y)\bar{j} + (3x-2y)\bar{k}$

Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} \Rightarrow d\bar{r} = dx\bar{i}+dy\bar{j}+dz\bar{k}$

$d\bar{r} = dx\bar{i}+dy\bar{j}$

$\therefore \bar{F}\cdot d\bar{r} = (2x-y)\,dx + (x+y)\,dy$.

(i). Now workdone $= \int_C \bar{F}\cdot d\bar{r}$, where C is the circle $x^2+y^2=9$

Take $x = 3\cos\theta \rightarrow dx = -3\sin\theta\, d\theta$

$y = 3\sin\theta \Rightarrow dy = 3\cos\theta\, d\theta$. & $\theta \rightarrow 0 \rightarrow 2\pi$

$$= \int_0^{2\pi} \left[(6\cos\theta - 3\sin\theta)(-3\sin\theta)\,d\theta + (3\cos\theta+3\sin\theta)\,3\cos\theta\, d\theta\right]$$

$$= \int_0^{2\pi} \left[(-18\cos\theta\sin\theta + 9\sin^2\theta)\,d\theta + (9\cos^2\theta + 9\sin\theta\cos\theta)\right] d\theta$$

$$= \int_0^{2\pi} \left[-9\sin\theta\cos\theta\, d\theta + \int_0^{2\pi} 9(\sin^2\theta+\cos^2\theta)\,d\theta\right]$$

$$= \int_0^{2\pi} \frac{-18\sin 2\theta}{2}\,d\theta + \int_0^{2\pi} 9\, d\theta$$

$= 18\pi$ //

* Find the workdone in moving a particle in the force field $\bar{F} = (3x^2)\bar{i} + (2zx-y)\bar{j} + z\bar{k}$, along the curve defined by $x^2=4y$, $3x^3=8z$ from $x=0 \rightarrow x=2$.

[A ...]

(6). Find the work done by a force $F = (x^2 - y^2)i + (x^2 + y^2)j$ which moves a particle in xy-plane from $(0,0)$ to $(1,1)$ along the parabola $y^2 = x$.

(7). If $F = (x^2 - 27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$, evaluate $\int_C F \cdot d\bar{r}$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the st.line from $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$.

Sol: Given $\bar{F} = (x^2 - 27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\bar{F} \cdot d\bar{r} = (x^2 - 27)\,dx - (6yz)\,dy + 8xz^2\,dz$$

(i). Along the st.line from $O = (0,0,0)$ to $A = (1,0,0)$

& 'x' varies from $0 \to 1$ Here $y = 0$ & $z = 0$, $dy = 0$ $dz = 0$

$$\therefore \int_{OA} \bar{F} \cdot d\bar{r} = \int_0^1 (x^2 - 27)\,dx = \left(\frac{x^3}{3} - 27x\right)_0^1 = \frac{1}{3} - 27 = -80/3.$$

(ii). Along the st.line from $A = (1,0,0)$ to $B = (1,1,0)$ $\frac{y-0}{1-0}$

'y' vary from $0 \to 1$. $z = 0$, $x = 1$; $dz = 0$; $dx = 0$.

$$\therefore \int_{AB} \bar{F} \cdot d\bar{r} = \int_0^1 -(6yz)\,dy = \left[-\cancel{6}^{3} z \frac{y^2}{\cancel{2}}\right]_0^1 = \left(-3zy^2\right)_0^1 = 0 \,//$$

(iii). Along the st.line from $B = (1,1,0)$ to $C = (1,1,1)$

$z: 0 \to 1$ Here $x = 1$, $y = 1$, $dx = 0$, $dy = 0$

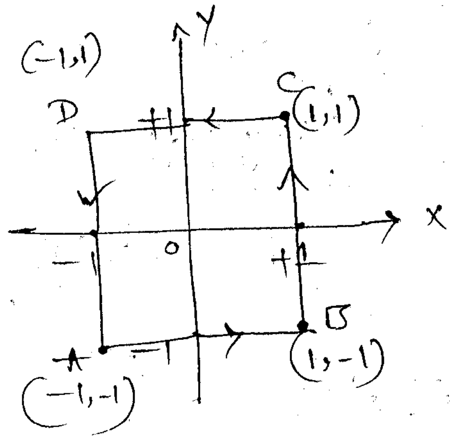
$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_0^1 8xz^2\,dz = \left(\frac{8z^3}{3}\right)_0^1 = \frac{8}{3} \,//$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BC} \bar{F} \cdot d\bar{r}$$

$$= -72/3 \,//$$

8). Evaluate the line integral $\int_C [(x^2+xy)\,dx + (x^2+y^2)\,dy]$ where 'C' is the square formed by the lines $x = \pm 1$ & $y = \pm 1$

Sol^n:-



Here $\int_C \bar{F}\cdot d\bar{r} = \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

from the diagram,

$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r} \quad -(1)$$

Along AB :- $A = (-1,-1)$, $B = (1,-1)$

Here $y = -1 \Rightarrow dy = 0$

$\therefore x : -1 \to 1$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (x^2+xy)\,dx = \int_{-1}^{1} (x^2-x)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_{-1}^{1} = \left(\frac{1}{3}-\frac{1}{2}\right) - \left(\frac{-1}{3}-\frac{1}{2}\right)$$

$$= \frac{1}{3}+\frac{1}{3} = 2/3.$$

Along BC :- $B = (1,-1)$, $C = (1,1)$

Here $x = 1 \Rightarrow dx = 0$

$y : -1 \to 1$

$$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_{-1}^{1} (x^2+y^2)\,dy = 2\int_0^1 (1+y^2)\,dy = 2\left(y+\frac{y^3}{3}\right)_0^1 = 2\left(1+\frac{1}{3}\right) = \frac{8}{3}$$

Along CD :- $C = (1,1)$, $D = (-1,1)$

Here $y = 1 \Rightarrow dy = 0$

$x : 1 \to -1$

$$\therefore \int_{CD} \bar{F}\cdot d\bar{r} = -\int_{-1}^{+1} (x^2+x)\,dx = -2\left(x^3/3+\frac{x^2}{2}\right)_0^1 = (-1)\left[\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\right]_{-1}^{1} = (-1)\left[\left(\frac{1}{3}+\frac{1}{2}\right)-\left(\frac{-1}{3}+\frac{1}{2}\right)\right]$$

$$= -1\left(\frac{2}{3}\right)$$

$$= -\left[\frac{x^3}{3}+\frac{x^2}{2}\right] = -\left[\frac{4}{6}\right] = -2/3$$

$= -2/3$

Along the curve DA :- $D=(-1,1)$ & $A=(-1,-1)$

$x=-1, dx=0$

$y: 1 \to -1$

$$\therefore \int_{DA} \bar{F}\cdot d\bar{r} = \int_{1}^{-1} (1+y^2)\,dy = (-1)\int_{-1}^{1}(1+y^2)\,dy = (-1)\left(y+\frac{y^3}{3}\right)_{-1}^{1}$$

$$= (-1)\left[\left(1+\frac{1}{3}\right)-\left(-1-\frac{1}{3}\right)\right]$$

$$= (-1)\left(2+\frac{2}{3}\right) = -8/3.$$

$$\therefore (1) \Rightarrow \int_C \bar{F}\cdot d\bar{r} = \cancel{\frac{2}{3}}+\cancel{\frac{8}{3}}-\cancel{\frac{2}{3}}-\cancel{\frac{8}{3}}$$

$$= 0 = \cancel{0}$$

$\boxed{\text{Note}}$: ① If '$\bar{V}$' represent a velocity of a fluid particle and 'c' is a closed curve, then the integral $\oint_C \bar{V}\cdot d\bar{r}$ is called the "Circulation" of '$\bar{V}$' around the curve 'C'. $\int_C \bar{F}\cdot d\bar{r}$

② If $\int_C \bar{V}\cdot d\bar{r} = 0$ ($\int_C \bar{F}\cdot d\bar{r} = 0$), then the field '$\bar{V}$' is called "Conservative" i.e. no work is done and the energy is conserved.

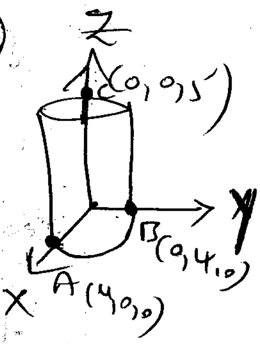
③ If the circulation of '$\bar{V}$' around every closed curve in a region 'D' vanishes, then '$\bar{V}$' is said to be "irrotational" in 'D'.

(1). Evaluate $\int \bar{F}\cdot\bar{n}\,ds$, where $F = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the 1^{st} octant between $z=0$ and $z=5$

Sol:- Given $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$ ——— (1)

& $\phi = x^2+y^2-16=0$. ——— (2)

Now the normal to the surface 'S' is grad ϕ is $\nabla\phi$



$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2x) + \bar{j}(2y)$$

Now the unit normal vector $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{x\bar{i}+y\bar{j}}{4}$

Let the projection is on the surface 'S' is in yz-plane then

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_S \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} \quad \text{——— (3)}$$

Here $\bar{F}\cdot\bar{n} = (z\bar{i} + x\bar{j} - 3y^2z\bar{k})\cdot\left(\frac{x\bar{i}}{4} + \frac{y\bar{j}}{4}\right)$

$$= \frac{zx}{4} + \frac{xy}{4} = \frac{xz+xy}{4}$$

$[\bar{n}\cdot\bar{i}] = \left[\frac{x}{4}\bar{i} + \frac{y}{4}\bar{j}\right]\cdot\bar{i} = \frac{x}{4}$

Since the surface $x^2+y^2=16$, which is in yz-plane i.e $x=0$

$$y = 4$$

Here $\therefore y: 0 \to 4$

Given $z: 0 \to 5$

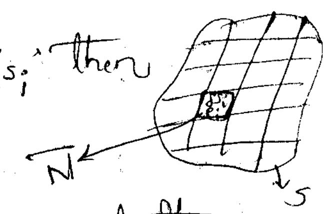
(3) $\Rightarrow \iint_S \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{5}\int_{y=0}^{4}\left(\frac{xz}{4} + \frac{xy}{4}\right)\frac{dy\,dz}{\left(\frac{x}{4}\right)}$

$$= \int_{z=0}^{5}\int_{y=0}^{4}\left(\frac{z+y}{\cancel{4}}\right) dy\,dz\,(\cancel{4})$$

$$= \int_0^5\int_0^4 (z+y)\,dy\,dz = \int_{z=0}^{5}\left(zy + \frac{y^2}{2}\right)_0^4 dz = \int_{z=0}^{5}(4z+8)\,dz = (4z^2+8z)_0^5 = 90$$

Surface Integrals :-

Let "$\overline{F}(\overline{r})$" be a continuous vector f^n defined on the smooth surface $\overline{r} = \overline{f}(u,v)$ and 's' be the region of the surface divided into 'm' sub regions of areas $\delta s_1, \delta s_2, \delta s_3, \dots, \delta s_m$ and "p_i" be the points on 's_i' and 'N_i' be the unit normal to 'δs_i' then



$\delta A_i = N_i \cdot \delta s_i$

Now the total area of the surface of $F(\overline{r})$ of the region 'S' of the given surface is $\int_S \overline{F}(\overline{r}) \cdot d\overline{A}$ (or) $\int_S \overline{F} \cdot \overline{N}\, ds$.

Cartesian form :- Let $\overline{F}(\overline{r}) = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$ is a continuous differentiable vector f^n of x, y, z.

$\cos\alpha, \cos\beta, \cos\gamma$ be the direction cosines of unit normal '$\overline{N}$'

$$\Rightarrow \overline{N} = \overline{i}\cos\alpha + \overline{j}\cos\beta + \overline{k}\cos\gamma$$

$$\therefore \overline{F}\cdot\overline{N} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

then $\int_S \overline{F}\cdot\overline{N}\, ds = \iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy$.

$\boxed{\text{Note}}$: 1) If the projection of the surface 's' is on XY-plane then

$$\iint_S \overline{F}\cdot\overline{N}\, ds = \iint_S \overline{F}\cdot\overline{N}\, \frac{dx\, dy}{|\overline{N}\cdot\overline{k}|} \quad \text{———} \quad (1)$$

2). the projection of the surface 's' is on YZ-plane then

$$\iint_S \overline{F}\cdot\overline{N}\, ds = \iint_S \overline{F}\cdot\overline{N}\, \frac{dy\, dz}{|\overline{N}\cdot\overline{i}|} \quad \text{———} \quad (2)$$

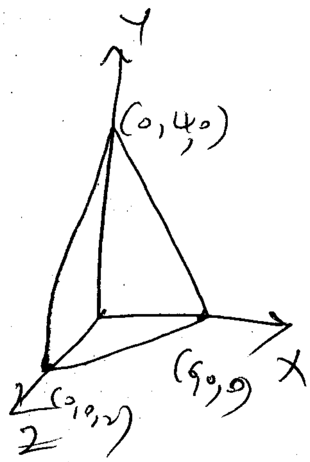
3). the projection of the surface 's' is on ZX-plane then

$$\iint_S \overline{F}\cdot\overline{N}\, ds = \iint_S \overline{F}\cdot\overline{N}\, \frac{dz\, dx}{|\overline{N}\cdot\overline{j}|} \quad \text{———} \quad (3)$$

(2). Evaluate $\iint_S \bar{F}\cdot d\bar{s}$ if $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2+y^2=9$ contained the 1^{st} octant between the planes $z=0$ and $z=2$. (Ans -18)

(3). Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$, where S is the surface of the cylinder $x^2+y^2=1$ in the 1^{st} octant between $z=0$, $z=2$. (3).

(4). Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the surface of the plane $2x+3y+6z=12$ located in the 1^{st} octant.



Sol:- Let $\phi = 2x+3y+6z-12=0$

Now the normal to the surface S is grad ϕ i.e. $\nabla\phi$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2) + \bar{j}(3) + \bar{k}(6) = 2\bar{i}+3\bar{j}+6\bar{k}$$

Now the unit normal vector $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{4+9+36}} = \frac{2}{7}\bar{i}+\frac{3}{7}\bar{j}+\frac{6}{7}\bar{k}$

Let us assume that the projection is xy-plane

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_S \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} \longrightarrow (1)$$

Here $\bar{F}\cdot\bar{n} = \frac{36z}{7} - \frac{36}{7} + \frac{18y}{7}$

$\bar{n}\cdot\bar{k} = \frac{6}{7}$

$2x+3y+6z=12$ is on xy-plane $\Rightarrow z=0$

$\Rightarrow 2x+3y=12$

$3y = 12-2x \Rightarrow y = \frac{12-2x}{3}$ $\therefore y: 0 \to \frac{12-2x}{3}$

If $y=0 \Rightarrow 2x=12$

$x=6$

$\therefore x: 0 \to 6.$

(12)

(1) $\Rightarrow \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_S \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{(\bar{n}\cdot\bar{k})}$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \left(\frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}\right)\frac{dx\,dy}{6/7}$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \frac{18}{7}\left[2z-2+y\right] dx\,dy \times \frac{7}{6}$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} 3\,(2z-2+y)\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6z-6+3y)\,dx\,dy$$

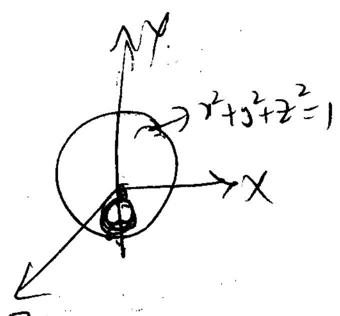
$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \left\{\cancel{6}\left(\frac{12-2x-3y}{\cancel{6}}\right)-6+3y\right\} dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{12-2x/3} (12-2x-6)\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dx\,dy$$

$$= \int_{x=0}^{6} \left(6y-2xy\right)_0^{12-2x/3} dx = 24 \,//$$

(5). If $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$, evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ over the surface $x^2+y^2+z^2=1$ in the 1st octant.



Sol:- Given $\phi = x^2+y^2+z^2=1$

$\nabla\phi = 2x\bar{i}+2y\bar{j}+2z\bar{k}$

$\bar{n} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\bar{i}+y\bar{j}+z\bar{k}$

Let the projection of on the surface 's' is in yz-plane $\Rightarrow x=0$.

$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_S \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$ ——— (1) & since $x^2+y^2+z^2=1$

$z = \sqrt{1-y^2}$

Here $\bar{F}\cdot\bar{n} = yzx + zxy + zxy = 3xyz$.   If $z=0 \Rightarrow y=1$

$\therefore (1) \Rightarrow \iint_S \bar{F}\cdot\bar{n}\, ds = \iint_S 3\cancel{x}yz \; dy\, dz$

$$= 3\int_0^1 \int_0^{\sqrt{1-y^2}} (yz)\; dy\, dz$$

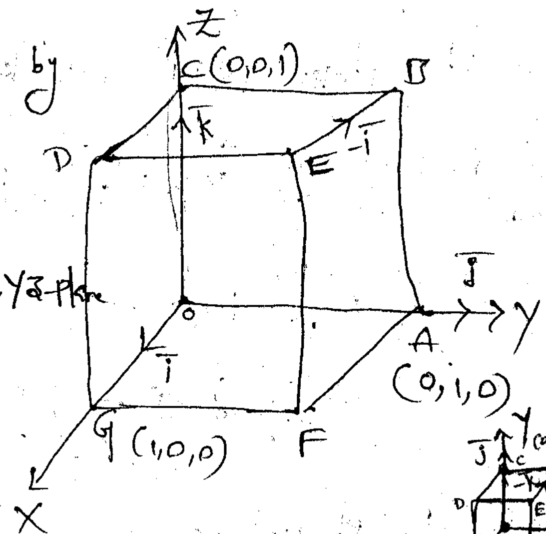
$$= 3\int_0^1 y\left(\frac{z^2}{2}\right)_0^{\sqrt{1-y^2}} dy$$

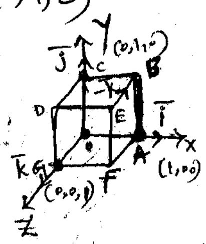
$$= 3\left[\int_0^1 y\left(\frac{1-y^2}{2}\right)\right] dy$$

$$= \frac{3}{2}\int_0^1 (y - y^3)\, dy = \frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)_0^1 = 3/8$$

(6). If $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$, Evaluate $\iint_S \bar{F}\cdot\bar{n}\, ds$ where 'S' is the surface of the cube bounded by $x=0, x=1;\ z=0, z=1;\ y=0, y=1$

Sol: Consider the cube surrounded by the following phases.





$\Rightarrow$ For the phase DEFG $\bar{n} = \bar{i}$, $x=1$ in yz-plane

$$\iint_{DEFG} \bar{F}\cdot\bar{n}\, ds = \iint (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot\bar{i}\ \frac{dy\, dz}{|\bar{n}\cdot\bar{i}|}$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} 4xz\; dy\, dz$$

$$= \int_{y=0}^{1}\left\{\int_{z=0}^{1} 4z\, dz\right\} dy = \int_{y=0}^{1}\left(\cancel{4}^{2}\,\frac{z^2}{\cancel{2}}\right)_0^1 dy = \int_{y=0}^{1} 2\, dy = (2y)_0^1 = 2$$

$\rightarrow$ For the phase OABC, $\bar{n} = -\bar{i}$, $x=0$ then

$$\iint_{OABC} \bar{F}\cdot\bar{n}\, ds = \int_0^1\int_0^1 (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot(-\bar{i})\; dy\, dz$$

$$= \int_0^1\int_0^1 -4xz\; dy\, dz$$

$$= \int_0^1 0$$

$$= 0$$

→ For the phase ABEF, $\bar{n}=\bar{j}$, $y=1$, then

$$\iint_{ABEF} \bar{F}\cdot\bar{n}\,ds = \iint (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot\bar{j}\,dx\,dz$$

$$= \iint (-y^2)\,dx\,dz$$

$$= \int_0^1\int_0^1 (-1)\,dx\,dz = \int_0^1 (-z)_0^1\,dx = -1$$

→ For the phase OCDG, $\bar{n}=-\bar{j}$, $y=0$, then

$$\iint_{OCDG} \bar{F}\cdot\bar{n}\,ds = \iint (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot(-\bar{j})\,dx\,dz$$

$$= \iint y^2\,dx\,dz$$

$$= 0.$$

→ For the phase BCDE, $\bar{n}=\bar{k}$, $z=1$ then

$$\iint_{BCDE} \bar{F}\cdot\bar{n}\,ds = \iint (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot\bar{k}\,dx\,dy$$

$$= \int_0^1\int_0^1 yz\,dx\,dy$$

$$= \int_0^1\int_0^1 (y\,dy)\,dx = \int_0^1 \left(\frac{y^2}{2}\right)_0^1 dx = \left(\frac{1}{2}x\right)_0^1 = \frac{1}{2}$$

→ For the phase OAGF, $\bar{n}=-\bar{k}$, $z=0$ then

$$\iint_{OAGF} \bar{F}\cdot\bar{n}\,ds = \iint (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot(-\bar{k})\,dx\,dy$$

$$= \iint -yz\,dx\,dy$$

$$= 0$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{DEFG} \bar{F}\cdot\bar{n}\,ds + \iint_{OABC} \bar{F}\cdot\bar{n}\,ds + \iint_{ABEF} \bar{F}\cdot\bar{n}\,ds +$$

$$\iint_{OCDG} \bar{F}\cdot\bar{n}\,ds + \iint_{BCDE} \bar{F}\cdot\bar{n}\,ds + \iint_{OAGF} \bar{F}\cdot\bar{n}\,ds$$

$$= 2+0-1+0+\tfrac{1}{2}+0 = \tfrac{3}{2}.$$

7. If $F = (4xz)\bar{i} - y^2\bar{j} + (yz)\bar{k}$, evaluate $\iint \bar{F}\cdot\bar{n}\, ds$ where 'S' is the surface of cube bounded by $x=0, x=a$; $y=0, y=a$; $z=0, z=a$. $\left(\frac{3a^4}{2}\right)$

8. If $\bar{F} = (x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$; evaluate $\iint \bar{F}\cdot\bar{n}\, ds$, where 'S' is the surface of the plane $2x+y+2z=6$ in the 1st octant (81)

* Volume Integrals *

Let $\bar{V}$ be a volume bounded by a surface $\bar{r} = \bar{f}(u,v)$ be a vector point f^n defined as $\boxed{\int_V \bar{F}(\bar{r})\, dv \text{ (or) } \int_V \bar{F}\, d\bar{v}}$

Cartesian form :– Let $\bar{F}(\bar{r}) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2, F_3 are f's of x, y, z and $d\bar{v} = dx\, dy\, dz$

$\Rightarrow \bar{F}\, d\bar{v} = (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\, dx\, dy\, dz$

$= F_1\, dx\, dy\, dz\, \bar{i} + F_2\, dx\, dy\, dz\, \bar{j} + F_3\, dx\, dy\, dz\, \bar{k}$

$\Rightarrow \int \bar{F}\, d\bar{v} = \iiint F_1\, dx\, dy\, dz\, \bar{i} + F_2\, dx\, dy\, dz\, \bar{j} + F_3\, dx\, dy\, dz\, \bar{k}$.

Problems :–

1. If $\bar{F} = (2xz)\bar{i} - x\bar{j} + y^2\bar{k}$, Evaluate $\iiint_V \bar{F}\, dv$, where 'V' is the region bounded by the Surfaces $x=0, y=0, y=6, z=x^2, z=4$.

Sol^n: Given $\bar{F} = (2xz)\bar{i} - x\bar{j} + y^2\bar{k}$ then the volume integral

$\int_V \bar{F}\, dv = \iiint (2xz\bar{i} - x\bar{j} + y^2\bar{k})\, dx\, dy\, dz$ ——— (1)

Given $z = x^2$, $z=4$ $\Rightarrow x^2 = 4$

$x = 2$

here $x: 0 \to 2$

$y: 0 \to 6$

$z: x^2 \to 4$

$$(1) \Rightarrow \int_V F\,dv = i\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,dx\,dy\,dz - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} \cancel{2}x\left(\frac{z^2}{\cancel{2}}\right)_{x^2}^{4} dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} x(z)_{x^2}^{4}\,dx\,dy + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2(z)_{x^2}^{4}\,dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} x(16-x^4)\,dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} x(4-x^2)\,dx\,dy + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2(4-x^2)\,dx$$

$$= \bar{i}\int_0^2\left[16xy - x^5y\right]_0^6 dx - \bar{j}\int_0^2\left(4xy - x^3y\right)_0^6 dx + \bar{k}\int_0^2\left[\frac{4y^3}{3} - \frac{x^2y^3}{3}\right]_0^6 dx$$

$$= 6\bar{i}\int_0^2(16x - x^5)\,dx - 6\bar{j}\int_0^2(4x - x^3)\,dx + \frac{6^3}{3}\bar{k}\int_0^2(4-x^2)\,dx$$

$$= 6\bar{i}\left(\frac{\cancel{16}^{8}x^2}{\cancel{2}} - \frac{x^6}{6}\right)_0^2 - 6\bar{j}\left(\frac{\cancel{4}^{2}x^2}{\cancel{2}} - \frac{x^4}{4}\right)_0^2 + 72\bar{k}\left(4x - \frac{x^3}{3}\right)_0^2$$

$$= 6\bar{i}\left\{8(4) - \frac{\cancel{64}^{32}}{\cancel{6}_3}\right\} - 6\bar{j}\left\{2(4) - \frac{\cancel{16}^{4}}{\cancel{4}}\right\} + 72\bar{k}\left\{4(2) - \frac{8}{3}\right\}$$

A.

14

$$= 6\bar{i}\left\{32 - \frac{32}{3}\right\} - 6\bar{j}\left\{8 - 4\right\} + 72\bar{k}\left\{8 - \frac{8}{3}\right\}$$

$$= 6\bar{i}\left\{\frac{96-32}{3}\right\} - 6\bar{j}\{4\} + 72\bar{k}\left\{\frac{24-8}{3}\right\}$$

$$= \cancel{6}^{2}\bar{i}\left(\frac{64}{\cancel{3}}\right) - 24\bar{j} + \cancel{72}^{24}\bar{k}\left(\frac{16}{\cancel{3}}\right)$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k}$$

$(8/3\,\bar{j} - \frac{5}{3}\bar{k})$

(2). $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$, evaluate $\int \nabla\cdot F\,dv$ & $\int \nabla\times F\,dv$ where 'V' is the closed region bounded by the planes $x=0, y=0, z=0$, $2x+2y+z=4$.

Ans. $\boxed{\nabla\cdot\bar{F} = 2x}$. $z=0 \Rightarrow 2x+2y=4$

$z: 0 \to 4-2x-2y$ | $2y = 4-2x$, $y = 2-x$, $\therefore y: 0 \to 2-x$ | If $x=0, y=0 \Rightarrow 2x=4$, $x=2$, $\therefore x = 0 \to 2$

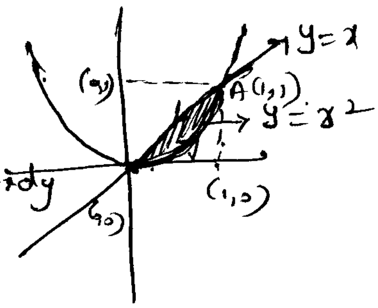
$\therefore \int \nabla\cdot\bar{F}\,dv = \frac{8}{3}$

$\int \nabla\times\bar{F}\,dv =$

* Green's theorem Extra Problems *

①. Verify Green's theorem $\int_C (xy+y^2)\,dx + x^2\,dy$, where 'C' is bounded by $y=x$, $y=x^2$

Ans: By the Green's theorem



w.k.T $\int_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

Given $y=x$ and $y=x^2$

intersect at $O=(0,0)$.

Intersection between $y=x$, $y=x^2$

$x=x^2$

$x=0,1$

$x: 0 \to 1$

$y: x^2 \to x$.

$M = xy+y^2$; $N=x^2$.

$\frac{\partial M}{\partial y} = x+2y$; $\frac{\partial N}{\partial x} = 2x$.

(i). L.H.S $= \int_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AO} M\,dx+N\,dy$ — (1).

along OA :- $O=(0,0)$
$A=(1,1)$

If we considering 'OA'

$y=x^2$ and $x: 0\to 1$

$dy = 2x\,dx$

$\therefore \int_{OA} M\,dx+N\,dy = \int_0^1 (yx+y^2)\,dx + (x^2)\,dy$

$= \int_0^1 \left[x(x^2)+(x^2)^2\right] dx + x^2(2x\,dx) = \frac{19}{20}$

along AO : $O=(0,0)$, $A=(1,1)$, consider $y=x$ & $x: 1 \to 0$
$dy = dx$

$$\int_{AO} M\,dx + N\,dy = \int_1^0 (xy+y^2)\,dx + x^2\,dy$$

$$= \int_1^0 (x^2+x^2)\,dx + x^2\,dx$$

$$= -1$$

$$(1) \Rightarrow \int_C M\,dx + N\,dy = -1 + \frac{19}{20} = -\frac{1}{20}.$$

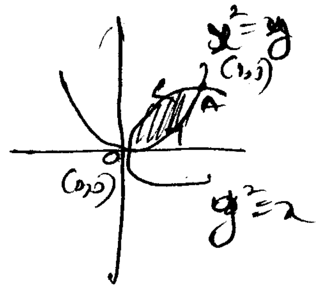
$$R.H.S = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= -\frac{1}{20}$$

Stoke's thm extra Problems

① Verify stoke's theorem for $\bar{F} = (2xy - x^2)\bar{i} - (x^2 - y^2)\bar{j}$, where 'c' boundary of region enclosed by parabola $y^2 = x$ and $y = x^2$?

Sol: By stoke's theorem we have to P.T



$$\int_C \bar{F}\cdot d\bar{r} = \int_S \operatorname{curl} \bar{F}\cdot \bar{n}\,ds \quad \text{— (1)}$$

R.H.S:

$$\operatorname{curl} \bar{F} = \nabla \times \bar{F} = -4x\bar{k}$$

Here 'S' is the surface in XY-plane.

$$\bar{n} = \bar{k}$$

$x: 0 \to 1$
$y: x^2 \to \sqrt{x}$.

$$\therefore \int \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} -4x\bar{k}\cdot\bar{k}\,dx\,dy$$

$$= -4\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} x\,dy\,dx = -4\int_{x=0}^{1} x\,(y)_{x^2}^{\sqrt{x}}\,dx$$

$$= -4\int_{x=0}^{1} x\,(\sqrt{x}-x^2)\,dx = -4\left(\frac{x^{5/2}}{5/2} - \frac{x^4}{4}\right)_0^1$$

$$= -4\left(\frac{2}{5}-\frac{1}{4}\right) = \frac{-\cancel{4}}{\cancel{20}_{5}}(3) = -3/5$$

L.H.S:- $\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AO} \bar{F}\cdot d\bar{r}$

Along OA :- i.e. $y = x^2$; $x: 0 \to 1$

$dy = 2x\,dx$

$$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = \int_{x=0}^{1}\left[(2xy - x^2)\bar{i} - (x^2-y^2)\bar{j}\right]\cdot\left[dx\bar{i} + dy\bar{j}\right]$$

$$= \int_{x=0}^{1} (2xy - x^2)\,dx - (x^2-y^2)\,dy$$

$$= 0$$

Along AO :- i.e. $y^2 = x$ and $y: 1 \to 0$.

$2y\,dy = dx$.

* Gauss divergence theorem *

①. Verify Gauss divergence theorem for $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ and 'S' is the surface of the cube, which is bounded by

$$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1.$$

Sol: $3/2$.

② Verify divergence thm for $\bar{f} = z\bar{i} + x\bar{j} + (-3y^2z)\bar{k}$ and 'S' is the surface of the cylinder $x^2+y^2=16$ b/w $z=0$ & $z=5$.

Sol:
$z: 0 \to 5$
$x: -4 \to 4$
$y: -\sqrt{16-x^2} \to \sqrt{16-x^2}$

(90π).

③ By Transforming to Evaluate $\iint x^3 dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where 'S' is the closed surface consisting of the cylinder $x^2+y^2=a^2$ and the circular disk $z=0$ & $z=b$.

Sol:
$z: 0 \to b$
$x: -a \to a$
$y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$

By G.D.T

$$\iint_S f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} 5x^2\,dx\,dy\,dz$$

$$= 5\cdot 2\cdot 2\int_{x=0}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 5x^2 (z)_0^b\,dx\,dy$$

$$\left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\right.$$

$$= 2ab\int_{x=0}^{a}\left[x^2(y)_0^{\sqrt{a^2-x^2}}\right]dx$$

$$= 2ab\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

$$= 2ab\int_{\theta=0}^{\pi/2} a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$

$x = a\sin\theta$
$dx = a\cos\theta\,d\theta$
Put $x=0 \Rightarrow \theta=0$
$x=a \Rightarrow \theta=\pi/2$

$$= 2ab\int_{\theta=0}^{\pi/2} a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$

$$= \frac{20ba^4}{4}\int_{\theta=0}^{\pi/2} 4\sin^2\theta\cos^2\theta\,d\theta$$

$$= 5a^4b\int_{\theta=0}^{\pi/2}\sin^2(2\theta)\,d\theta$$

$$= 5a^4b\int_{\theta=0}^{\pi/2}\left[\frac{1-\cos4\theta}{2}\right]d\theta$$

$$= \frac{5}{2}a^4b\left[\theta-\frac{\sin4\theta}{4}\right]_{\theta=0}^{\pi/2}$$

$$= \frac{5}{2}a^4b\left[\frac{\pi}{2}-\frac{\sin4(\pi/2)}{4}-\frac{0+\sin4(0)}{4}\right]$$

$$= \frac{5}{4}\pi a^4b.$$

4

Assignment Questions

state stokes theorem and

1. Verify stokes theorem for $F = (2x-y)\bar{i} - (yz^2)\bar{j} - (y^2z)\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection on the xy-plane.

state Green's theorem and

2. Verify Green's theorem $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where 'C' is the region bounded by $y=\sqrt{x}$ and $y=x^2$.

state Gauss-divergence theorem and

3. Verify Gauss divergence theorem for $2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$.

4. a) Show that $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

   b) Show that $\dfrac{\bar{r}}{r^3}$ is solenoidal, where $r = |\bar{r}|$.

5. a) Evaluate $\iint_R y\,dx\,dy$, where 'R' is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$

   b) By changing into polar co-ordinates Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$

Gauss Divergence theorem *

(Transformation between surface and volume integral)

Statement :- Let 's' be a closed surface enclosing a volume 'V'. If '$\bar{F}$' is a continuously differentiable vector point f^n, then

$\boxed{\int_V \operatorname{div}\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds}$, where '$\bar{n}$' is the outward normal vector at any point of 's'.

(or)

$$\iiint\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \iint_S F_1\,dy\,dz + F_2\,dx\,dz + f_3\,dx\,dy$$

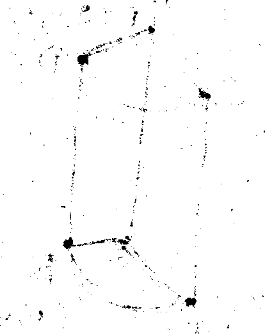
① Verify Divergence theorem for $2x^2y\bar{i}-y^2\bar{j}+4xz^2\bar{k}$ taken over the region of 1st octant of the cylinder $y^2+z^2=9$ and $x=2$

(or)

Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{F}=2x^2y\bar{i}-y^2\bar{j}+4xz^2\bar{k}$ and 's' is the closed surface of the region in the 1st octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0, x=2, y=0, z=0$.

Sol^n:- By the Gauss Divergence theorem

$$\int_V \operatorname{div}\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds \quad \text{———} \quad (1)$$



Given $\bar{F} = (2x^2y)\bar{i} - (y^2)\bar{j} + (4xz^2)\bar{k}$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(2x^2y)+\frac{\partial}{\partial y}(-y^2)+\frac{\partial}{\partial z}(4xz^2)$$

[illegible] $+ 8xz$

3. Vector Integral Theorems

Gauss Divergence theorem

(Transformation between surface and volume integral)

statement :- Let 's' be a closed surface enclosing a volume 'V'. If $\bar{F}$ is a continuously differentiable vector point f^n, then

$\boxed{\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds}$, where '$\bar{n}$' is the outward normal vector at any point of 's'.

(or)

$$\iiint \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S F_1\, dy\, dz + F_2\, dx\, dz + F_3\, dx\, dy$$

1. Verify Divergence theorem for $2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ taken over ~~the region of 1st octant of the cylinder $y^2+z^2=9$ and $x=2$~~

(or)

Evaluate $\iint_S \bar{F}\cdot\bar{n}\, ds$, where $\bar{F} = 2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ and 's' is the closed surface of the region in the 1st octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0, x=2, y=0, z=0$.

sol^n:- By the Gauss Divergence theorem

$$\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds \quad \text{—— (1)}$$

Given $\bar{F} = (2x^2y)\bar{i} - (y^2)\bar{j} + (4xz^2)\bar{k}$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$\iiint_V \nabla\cdot F\, dv = \int_{x=0}\int_{y=0}\int_{z=0} (4xy - 2y + 8xz)\, dz\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \left[4xyz - 2yz + \cancel{8}^{4} x \frac{z^2}{\cancel{2}}\right]_{z=0}^{z=\sqrt{9-y^2}} dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \left[(4xy - 2y)\sqrt{9-y^2} + 4x(9-y^2)\right] dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \left[(1-2x)(-2y)\sqrt{9-y^2} + 4x(9-y^2)\right] dy\, dx$$

$$= \int_{x=0}^{2} \left\{\left[(1-2x)\frac{(9-y^2)^{3/2}}{(3/2)}\right]_0^3 + 4x\left(9y - \frac{y^3}{3}\right)_0^3\right\} dx \qquad \left[\because \int f'(x)[f(x)]^n dx = \frac{(f(x))^{n+1}}{n+1} + c\right]$$

$$= \int_{x=0}^{2} \left\{\frac{2}{\cancel{3}}(1-2x)(0-\cancel{27}^{9}) + 4x(27-9)\right\} dx$$

$$= \int_{x=0}^{2} \{-18(1-2x) + 72x\}\, dx$$

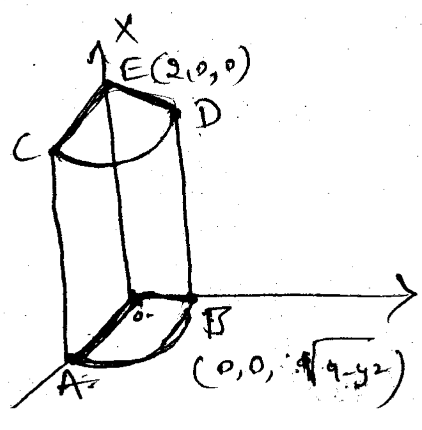
$$= \int_{x=0}^{2} \{-18 + 36x + 72x\}\, dx = \int_{x=0} (-18 + 108x)\, dx = \left(-18x + \cancel{108}^{54}\frac{x^2}{\cancel{2}}\right)_0^2$$

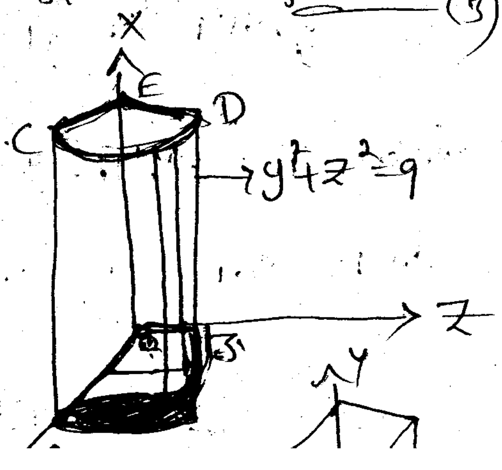
$$= -36 + 216$$

$$= 180 \quad \text{--- (2)}$$

Now find $\int_S \bar{F}\cdot\bar{n}\, ds$:-

$$\therefore \int_S \bar{F}\cdot\bar{n}\, ds = \int_{S_1} \bar{F}\cdot\bar{n}\, ds + \int_{S_2} \bar{F}\cdot\bar{n}\, ds + \int_{S_3} \bar{F}\cdot\bar{n}\, ds + \int_{S_4} \bar{F}\cdot\bar{n}\, ds + \int_{S_5} \bar{F}\cdot\bar{n}\, ds \quad \text{--- (3)}$$





(i). on S_1: OAB , Here $\bar{n} = -\bar{i}$, $x=0$, $ds = dy\,dz$.

$$\therefore \bar{F}\cdot\bar{n} = (2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot(-\bar{i})$$

$$= -2x^2y$$

$$= 0$$

$$\boxed{\therefore \int_{S_1} \bar{F}\cdot\bar{n}\,ds = 0}$$

(ii) on S_2: CED, Here $\bar{n} = \bar{i}$, $x=2$, $ds = dy\,dz$

$$\therefore \bar{F}\cdot\bar{n} = (2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot\bar{i}$$

$$= 2x^2y$$

$$= 8y$$

$$\therefore \int_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}} 8y\,dy\,dz.$$

$$= \int_{z=0}^{3}\left(\frac{8y^2}{2}\right)_0^{\sqrt{9-z^2}} dz = \int_{z=0}^{3}[4(9-z^2)]\,dz = 4\left(9z-\frac{z^3}{3}\right)_{z=0}^{3} = 72.$$

(iii). on S_3: OBDE , Here $\bar{n} = -\bar{j}$, $y=0$, $dx\,dz = ds$.

$$\therefore \bar{F}\cdot\bar{n} = (2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot(-\bar{j})$$

$$= y^2 = 0$$

$$\int_{S_3} \bar{F}\cdot\bar{n}\,ds = 0.$$

(iv). on S_4: CFOA , Here $\bar{n} = -\bar{k}$, $z=0$, $ds = dx\,dy$

$$\therefore \int \bar{F}\cdot\bar{n}\,ds = \int (2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot(-\bar{k})\,dx\,dy$$

$$= 0$$

(v) on S_5: ABCD ; Here $\phi = y^2 + z^2 - 9 = 0$

$$\therefore \iint_{S_5} \bar{F}\cdot\bar{n}\, ds = \iint_{S_5} \bar{F}\cdot\bar{n}\, \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \frac{-y^3+4xz^3}{\cancel{3}} \cdot \frac{dx\,dy}{\frac{1}{\cancel{3}}z}$$

$$= \int_0^2\int_0^3 \frac{(-y^3+4xz^3)}{z}\, dx\,dy$$

$$= \int_0^2\int_0^3 \left(4xz^2 - \frac{y^3}{z}\right) dx\,dy$$

$$= \int_0^2\left\{\int_0^3 4x(9-y^2) - y^3(9-y^2)^{-1/2}\, dy\right\} dx$$

$$= \int_0^2\left\{\int_0^3 4x(9-y^2) - y^3(9-y^2)^{-1/2}\, dy\right\} dx$$

$$= \int_0^2 4x\left(9y - \frac{y^3}{3}\right)_0^3 - \int_0^2\left\{\int_0^3 y^3(9-y^2)^{-1/2}\, dy\right\} dx$$

$$= \int_0^2 4x(27-9) - \int_0^2 18\, dx$$

$$= \left(72\frac{x^2}{2}\right)_0^2 - 18(x)_0^2$$

$$= 144 - 36$$

$$= 108.$$

$$\therefore (3) \Rightarrow \int_S \bar{F}\cdot\bar{n}\, ds = 0+72+0+0+108 = 180. \quad \text{———} (4)$$

$\therefore$ from (2) & (4),

$$\int_S \bar{F}\cdot\bar{n}\, ds = \int_V \nabla\cdot\bar{F}\, dv$$

1. Verify Gauss divergence theorem for $\bar{F} = x^2\bar{i}$, over the cube formed by the planes $x=0, x=a, y=0, y=b, z=0, z=c$. [$abc(a+b+c)$]

2. Verify Gauss divergence thm for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$, taken over the surface of the cube bounded by the planes $x=y=z=a$ and co-ordinate planes. $\left(\frac{a^5}{3}+a^3\right)$

4. Use Divergence theorem to evaluate $\iint_S (y^2z^2\bar{i} + z^2x^2\bar{j} + z^2y^2\bar{k})\cdot\bar{n}\,ds$. 'S' is the part of the unit sphere above xy-plane.

Soln Here $\bar{F} = y^2z^2\bar{i} + z^2x^2\bar{j} + z^2y^2\bar{k}$

By divergence thm $\iint_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$ —— (1)

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 2zy^2$$

$$(1) \Rightarrow \iint_S \bar{F}\cdot\bar{n}\,ds = \iiint 2zy^2\,dx\,dy\,dz$$

Introducing the spherical polar co-ordinates given by

Put $x = r\sin\theta\cos\phi \Rightarrow dx = -r\cos\theta\sin\phi\,d\theta\,d\phi$

$y = r\sin\theta\sin\phi \Rightarrow dy = r\cos\theta\cos\phi\,d\theta\,d\phi$

$z = r\cos\theta \Rightarrow dz = -r\sin\theta\,d\theta$.

$$dx\,dy\,dz = r^2\,dr\,d\theta\,d\phi.$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = 2\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)\,r^2\sin^2\theta\sin^2\phi\;r^2\,dr\,d\theta\,d\phi \quad [\because \cos 2\theta = 1-2\sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^5\cos\theta\sin^2\theta\left\{\frac{1-\cos 2\phi}{2}\right\}dr\,d\theta\,d\phi$$

Use G[redacted]nce theorem to evaluate $\iint_S (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}) \cdot ds$, where 'S' is the closed surface bounded by the xy-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.

Sol: From the Divergence thm

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$$

$$\therefore \iint \bar{F}\cdot\bar{n}\,ds = \iiint_V 4z\,dx\,dy\,dz$$

Introducing the spherical coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

then $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^3\cos\theta\sin\theta\,dr\,d\theta\,d\phi$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\cos\theta\sin\theta\,(\phi)_0^{2\pi}\,dr\,d\theta$$

$$= \frac{\cancel{4}}{\cancel{2}}\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\,2\cos\theta\sin\theta\,(2\pi-0)\,dr\,d\theta$$

$$= 4\pi\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3(\sin 2\theta)\,dr\,d\theta = 4\pi\int_{r=0}^{a} r^3\left[-\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\pi} dr$$

$$= 0$$

6. Use Divergence theorem to evaluate $\iint_S \bar{F}\cdot d\bar{s}$, where $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and 'S' is the surface of the sphere

$x^2+y^2+z^2 = a^2$

Q. Use Divergence theorem to evaluate $\iint_S \bar{F}\cdot d\bar{s}$ where $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and 'S' is the surface bounded by the region $x^2+y^2=4$, $z=0$ & $z=3$.

Sol: We have $\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$ ——— (1)

$$\operatorname{div}\bar{F} = \nabla\cdot F = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

$$(1) \Rightarrow \quad = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left((4-4y)z + z^2\right)_0^3 dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \{12(1-y)+9\}\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\,dx\,dy = \int_{-2}^{2}\left(21y - 12\frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\,dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\,dy\right] dx$$

$$= \int_{-2}^{2}\left[21\times 2\int_0^{\sqrt{4-x^2}} dy - 12(0)\right] dx$$ (∵ First term is even fn & 2nd " " odd)

$$= 42\int_{-2}^{2} (y)_0^{\sqrt{4-x^2}}\,dx$$

$$= 42\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 42\times 2\int_0^{2} \sqrt{4-x^2}\,dx$$

(3). Verify divergence thm for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ over the surface 'S' of the solid cut off by the plane $x+y+z=a$ in the first octant. $\left(\frac{a^4}{4}\right)$

(9)

Green's theorem :- (Transformation b/w line integral & double integral)

Statement:- If 'R' is a closed region in xy-plane bounded by a simple closed curve 'C' and if 'M' and 'N' are continuous f^{n}s of 'x' and 'y' having continuous derivatives in 'R', then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where 'C' is traversed in the +ve direction (anti-clockwise)

Problems:-

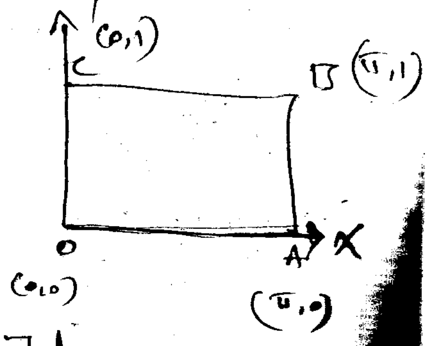
①. Evaluate by Green's theorem $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ where 'C' is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$, $(0,1)$.

Sol^n:- $M = x^2 - \cosh y$, $N = y + \sin x$

$\frac{\partial M}{\partial y} = -\sinh y$, $\frac{\partial N}{\partial x} = \cos x$

By Green's theorem $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\Rightarrow \int_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy = \iint_R (\cos x + \sinh y)\,dy\,dx$$



$$= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$$

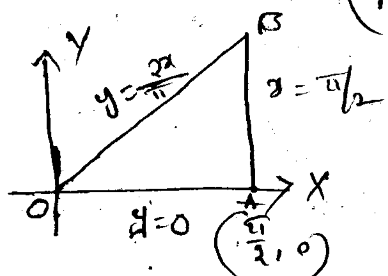
$$= \int_{x=0}^{\pi} \left((\cos x)y + \cosh y\right)_{y=0}^{1} dx = \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]\,dx$$

$$= (\sin x + x\cosh 1 - x)_0^{\pi} = \pi\cosh 1 - \pi = \pi(\cosh$$

(2). Evaluate by Green's thm $\oint_C (y-\sin x)\,dx + \cos x\,dy$, where 'c' is the triangle enclosed by the lines $x=\frac{\pi}{2}$, $y=0$, $\pi y = 2x$. $-\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$

$\left(x: 0 \to \pi/2 \;;\; y: 0 \to \frac{2x}{\pi}\right)$

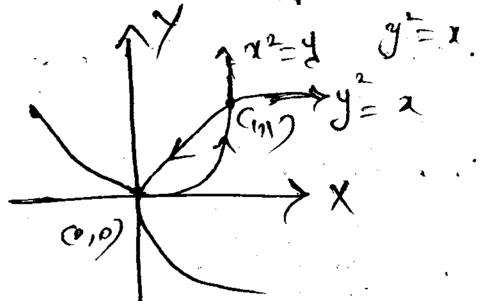
$\left[\text{Ans} = -\frac{\pi}{4}+\frac{2}{\pi}\right]$



(3). Using Green's thm evaluate $\int_C (2xy-x^2)\,dx + (x^2+y^2)\,dy$, where "C" is the closed curve of the region bounded by $y=x^2$ & $y^2=x$.

$\left(\begin{aligned} &y=x^2 \\ &y^2=x \Rightarrow y=\sqrt{x} \end{aligned} \Rightarrow x^2=\sqrt{x} \Rightarrow x^4-x=0\right.$

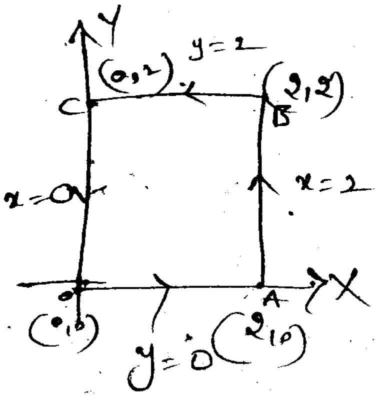
$\left.\therefore y: x^2 \to \sqrt{x} \;\&\; x: 0 \to 1\right)$



(4). Verify Green's thm in the plane for $\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$ where 'C' is a square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$

Ans:- Green's theorem states that

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—(1)}$$



Here $M = x^2 - xy^3$, $N = y^2 - 2xy$

$\frac{\partial M}{\partial y} = -3xy^2$, $\frac{\partial N}{\partial x} = -2y$

$\underline{\int M\,dx + N\,dy}$:-

(i). Along OA $(y=0)$

(ii). Along BC $(y=2)$

(iii). Along AB $(x=2)$

(iv). Along CO $(x=0)$

5. Find the circulation of $\bar{F}$ round the curve 'C' when $\bar{F} = (e^x \sin y)\bar{i} + (e^x \cos y)\bar{j}$ and 'C' is the rectangle whose vertices are $(0,0)$, $(1,0)$, $(1,\pi/2)$, $(0,\pi/2)$.

∴ Circulation of $\bar{F}$ round $C = \int_C \bar{F}\cdot d\bar{r} = \int_C (e^x \sin y\,\bar{i} + e^x \cos y\,\bar{j})(dx\bar{i} + dy\bar{j})$

$= \int_C e^x \sin y\, dx + e^x \cos y\, dy$ → which is in the form of

$\int_C M dx + N dy$, then $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Green's thm $\int_C M dx + N dy$

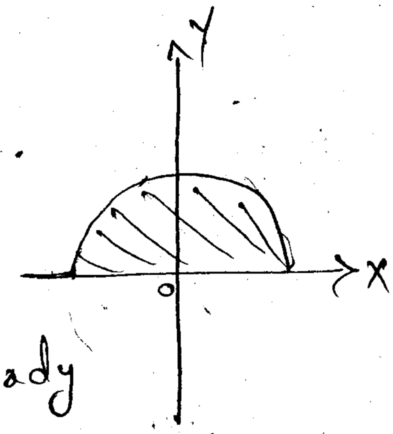
Here $M = e^x \sin y$ $\qquad = \iint_R e^x \cos y - e^x \cos y$

$\frac{\partial M}{\partial y} = e^x \cos y$ $\qquad = 0$ //

& $N = e^x \cos y$

$\frac{\partial N}{\partial x} = e^x \cos y$

6. Apply Green's thm to evaluate $\oint_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where 'C' is the boundary of the area enclosed by the x-axis and upper half the circle $x^2 + y^2 = a^2$.



Sol: $M = 2x^2 - y^2 \Rightarrow \frac{\partial M}{\partial y} = -2y$ & $N = x^2 + y^2$, $\frac{\partial N}{\partial x} = 2x$.

By Green's thm $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\Rightarrow \int_C (2x^2 - y^2)\,dx + (x^2 + y^2)dy = \iint_R (2x + 2y)\,dx\,dy = 2\iint_R (x + y)\,dx\,dy$

$= 2\int_0^a \int_0^\pi r(r\cos\theta + r\sin\theta)\,dr\,d\theta$ $\quad \left(\because x = r\cos\theta \Rightarrow y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta\right)$

$= 2\int_{r=0}^{a} r^2 dr \int_{\theta=0}^{\pi} (\cos\theta + \sin\theta)\, d\theta$

$= \ldots (\sin\theta - \cos\theta)_0^\pi = 2\left[\frac{r^3}{3}\right]_0^a (1+1) = \frac{4a^3}{3}$ //

(i). Along OA $(y=0)$:— $y=0 \Rightarrow dy=0$, $x: 0 \to 2$

$$\int_C (x^2-xy^3)dx+(y^2-2xy)dy = \int_{x=0}^{2} x^2\,dx = \left(\frac{x^3}{3}\right)_0^2 = 8/3.$$

(ii). Along AB :— $x=2$, $y: 0\to 2$
$dx=0$

$$\int_C (x^2-xy^3)dx+(y^2-2xy)dy = \int_{y=0}^{2}(y^2-2xy)\,dy = \int_{y=0}^{2}(y^2-4y)\,dy = \left(\frac{y^3}{3}-\frac{4y^2}{2}\right)_0^2$$

$$=\left(\frac{8}{3}-8\right)-0=\frac{-16}{3}$$

(iii) Along BC :— $y=2$, $x: 2\to 0$
$dy=0$

$$\int_C (x^2-xy^3)dx+(y^2-2xy)\,dy = \int_{x=2}^{0}(x^2-8x)\,dx = \left(\frac{x^3}{3}-\frac{8x^2}{2}\right)_2^0 = -\left(\frac{8}{3}+16\right)=\frac{40}{3}$$

(iv). Along CO :— $x=0$, $y: 2\to 0$
$dx=0$

$$\int_C (x^2-xy^3)dx+(y^2-2xy)\,dy = \int_{y=2}^{0}(y^2-2xy)\,dy = \left(\frac{y^3}{3}\right)_2^0 = -8/3$$

$$\therefore \int_C (x^2-xy^3)\cancel{dx}+(\cancel{y^2-2xy})\,dy = \frac{8}{3}-\frac{16}{3}+\frac{40}{3}-\frac{8}{3}=\frac{24}{3}=8.$$

$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$:—

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \int_0^2\int_0^2 (-2y+3xy^2)\,dx\,dy$$

$$=\int_0^2 \left(-2yx+3y^2\frac{x^2}{2}\right)_0^2 dy$$

$$=\int_0^2 (-4y+6y^2)\,dy = \left(-4\frac{y^2}{2}+6\frac{y^3}{3}\right)_0^2 = \left(-2y^2+2y^3\right)_0^2 = 8.$$

$\therefore$ Hence Verified //

STOKE's theorem :-

(Transformation between line Integral and Surface integral)

statement :- Let 'S' be a open surface bounded by a closed, non intersecting curve 'C'. If '$\bar{F}$' is any differentiable vector point function then $\oint_C \bar{F}\cdot d\bar{r} = \int_S curl\bar{F}\cdot\bar{n}\,ds$, where 'C' is traversed in the positive direction and '$\bar{n}$' is unit outward drawn normal at any point of the surface.

①. Evaluate by stoke's theorem $\int_C (e^x dx + 2y\,dy - dz)$ where 'C' is the curve $x^2+y^2=9$ & $z=2$.

Sol:- Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $\bar{F}\cdot d\bar{r} = (F_1\bar{i}+F_2\bar{j}+F_3\bar{k})\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k})$

$= e^x dx + 2y\,dy - dz$

Here $F_1 = e^x$, $F_2 = 2y$, $F_3 = -1$

stoke's thm state that

$$\int_C \bar{F}\cdot d\bar{r} = \int_S curl\,\bar{F}\cdot\bar{n}\,ds \quad \text{—— (1)}$$

Here $curl\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(0-0)$

$= 0.$

$(1) \Rightarrow \int_C \bar{F}\cdot d\bar{r} = 0.$

2). Verify stokes theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ (14).
over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane

Sol:- Given $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$

$$x^2+y^2+z^2=1$$

$$\Rightarrow x^2+y^2=1 \quad (\because xy\text{-plane } z=0)$$

the parametric Eq's are put $x = \cos\theta$, $y = \sin\theta$

$$dx = -\sin\theta\, d\theta, \quad dy = \cos\theta\, d\theta.$$

stokes thm states that, $\int_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds$ —— (1)

L.H.S:-

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$$

$$= \int_0^{2\pi} F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \int_0^{2\pi} (2x-y)\,dx + (-yz^2)\,dy - (y^2z)\,dz$$

$$= \int_0^{2\pi} (2x-y)\,dx$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} -2\sin\theta\cos\theta\,d\theta + \int_0^{2\pi} \sin^2\theta\,d\theta$$

$$= -\int_0^{2\pi} \sin 2\theta\,d\theta + \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$\left(\cos 2\theta = 1 - 2\sin^2\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$\therefore = \left(\frac{\cos 2\theta}{2}\right)_{\theta=0}^{2\pi} + \frac{1}{2}(\theta)_0^{2\pi} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)_0^{2\pi}$

$= \left(\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{2}(2\pi) - 0 = \pi.$

R.H.S:- $\text{curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$= \bar{i}\left(-2y\cancel{z} + 2\cancel{z}y\right) - \bar{j}(0-0) + \bar{k}(1) = \bar{k}$

Here $\bar{n} = \bar{k}$ ($\because$ 'R' is the projection on xy-plane)

$\therefore \int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \iint \bar{k} \cdot \bar{k}\, dx\, dy$

$(\because x^2 + y^2 + z^2 = 1$

$x^2 + y^2 = 1$

$y = \sqrt{1-x^2}$

$y : -\sqrt{1-x^2} \to \sqrt{1-x^2}$

$x = -1 \to 1)$

$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\, dy$

$= \int_{-1}^{1} 2\left(\sqrt{1-x^2}\right) dx$

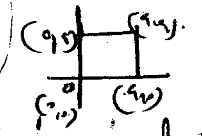
$= 4\left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right\}_0^1$ $\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c\right]$

$= 4\left(\frac{1}{2} \cdot \frac{\pi}{2}\right) = \cancel{4}\frac{\pi}{\cancel{4}}$

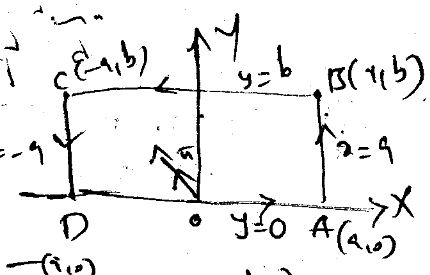
$= \pi$

$\therefore \int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds = \int_C \bar{F} \cdot d\bar{r}$

(3) HW:- Verify Stokes thm for the f^n $\bar{F} = x^2\bar{i} + xy\bar{j}$ integrated round the square in the plane $z=0$, whose sides are along the lines $x=0,\ y=0,\ x=a,\ y=a$. $\left(\frac{a^3}{2}\right)$



4. Verify Stokes thm for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$, taken round the rectangle bounded by the lines $x=\pm a$, $y=0$, $y=b$.



$$\int_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds \quad (-4ab^2)$$

(5). V.S.T $\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$, S is the surface of the cube bounded by $x=0, y=0, z=0, x=2, y=2, z=2$, above the xy plane. (-4)

(6). V.S.T $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes -

$x=0,\ x=a,\ y=0,\ y=b$. $(2ab^2)$